Combinatorics in Banach space theory

PROBLEMS (Part 4)*

● **PROBLEM 4.1.** Show that for every Banach space X and every $1 \leqslant p \leqslant \infty$ we have $\ell_p(X) \oplus \ell_p(X) \simeq X \oplus \ell_p(X) \simeq \ell_p(X)$ and $c_0(X) \oplus c_0(X) \simeq X \oplus c_0(X) \simeq c_0(X)$, where $\ell_p(X)$ and $c_0(X)$ stand for the ℓ_p- and c_0-sums of countably many copies of X (see Definition 5.1 from the lecture notes).

● **PROBLEM 4.2.** Let $\mathbb{D}$ be a set, $\mathbb{D}^*$ be the collection of all non-principal ultrafilters contained in $\mathcal{P}\mathbb{D}$, and for any $A \in \mathcal{P}\mathbb{D}$ let

$$A^* = \Big\{\mathsf{p} \subset \mathcal{P}\mathbb{D} \colon \mathsf{p} \text{ is a non-principal ultrafilter with } A \in \mathsf{p}\Big\}$$

and

$$\overline{A} = \Big\{\mathsf{p} \subset \mathcal{P}\mathbb{D} \colon \mathsf{p} \text{ is an ultrafilter with } A \in \mathsf{p}\Big\}.$$

We consider the topological spaces: $\mathbb{D} \cup \mathbb{D}^*$, with the topology generated by the basis $\{A \cup A^* \colon A \in \mathcal{P}\mathbb{D}\}$, and the Stone space $\mathsf{St}(\mathcal{P}\mathbb{D})$ consisting of all ultrafilters contained in $\mathcal{P}\mathbb{D}$, with the topology generated by the basis $\{\overline{A} \colon A \in \mathcal{P}\mathbb{D}\}$ (see Lecture 6 for further details). Show that the map $\varphi \colon \mathbb{D} \cup \mathbb{D}^* \to \mathsf{St}(\mathcal{P}\mathbb{D})$ given by

$$\begin{cases} \varphi(x) = \{A \subset \mathbb{D} \colon x \in A\}, & \text{for } x \in \mathbb{D} \\ \varphi(\mathsf{p}) = \mathsf{p}, & \text{for } \mathsf{p} \in \mathbb{D}^* \end{cases}$$

is a homeomorphism.

Remark. Both of these topological spaces may be considered as the definition of the Stone-Čech compactification $\beta\mathbb{D}$ of the discrete space $\mathbb{D}$.

● **PROBLEM 4.3.** It is to be proved that for any set $\mathbb{D}$ the remainder space $\beta\mathbb{D} \setminus \mathbb{D}$ is (homeomorphic to) the Stone space of the quotient Boolean algebra $\mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}$ ($\mathcal{F}\mathbb{D}$ stands for the ideal of all finite subsets of $\mathbb{D}$). More precisely, let $\mathsf{CO}(\beta\mathbb{D} \setminus \mathbb{D})$ be the algebra of all clopen subsets of $\beta\mathbb{D} \setminus \mathbb{D}$ and let $\psi \colon \mathsf{CO}(\beta\mathbb{D} \setminus \mathbb{D}) \to \mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}$ be given by $\psi(A^*) = [A]_{\mathcal{F}\mathbb{D}}$, where A^* is defined as in Problem 4.2 and $[A]_{\mathcal{F}\mathbb{D}}$ is the coset in $\mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}$ determined by A. Recall that every clopen subset of $\beta\mathbb{D} \setminus \mathbb{D}$ has the form A^* for some $A \in \mathcal{P}\mathbb{D}$ (see Lecture 6). Show that ψ yields an isomorphism between the algebras $\mathsf{CO}(\beta\mathbb{D} \setminus \mathbb{D})$ and $\mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}$.

● **PROBLEM 4.4.** Define a set algebra $\mathscr{F} \subset \mathcal{P}\mathbb{N}$ by

$$\mathscr{F} = \Big\{A \subset \mathbb{N} \colon |A \cap \{2k-1, 2k\}| \in \{0, 2\} \text{ for all but finitely many } k \in \mathbb{N}\Big\}.$$

For every $A \in \mathscr{F}$ choose any set $A' \in \mathscr{F}$ that contains almost all numbers from A and satisfies $|A' \cap \{2k-1, 2k\}| \in \{0, 2\}$ for each $k \in \mathbb{N}$. Now, for every non-principal ultrafilter $\mathsf{p} \in \mathsf{St}(\mathscr{F})$ let $\varphi(\mathsf{p})$ be an ultrafilter containing the sets A''s for all $A \in \mathsf{p}$. Show that $\varphi(\mathsf{p})$ is uniquely determined by p and that this definition does not depend on the choice of the sets A''s. Next, show that the so-defined map $\varphi \colon \mathsf{St}(\mathscr{F}) \setminus \mathbb{N} \to \beta\mathbb{N} \setminus \mathbb{N}$ (here we identify $\mathbb{N}$ with the set of all principal ultrafilters contained in $\mathscr{F}$, which is simply the set of all isolated points in $\mathsf{St}(\mathscr{F})$) is a homeomorphism.

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*Evaluation: ●=2pt, ●=3pt, ●=4pt

● **PROBLEM 4.5.** Let $\mathscr{F}$ be a set algebra and $B(\mathscr{F})$ be the Banach space of all scalar-valued functions that are uniform limits of sequences of $\mathscr{F}$-measurable step functions, equipped with the supremum norm. Prove that $B(\mathscr{F})$ is isometrically isomorphic to $C(\mathsf{St}(\mathscr{F}))$, the Banach space of all continuous functions on the Stone space of $\mathscr{F}$.

Hint. Note that characteristic functions of clopen sets are continuous. Use the Stone–Weierstrass theorem: If K is a compact Hausdorff space and $\mathcal{A}$ is a subalgebra of $C(K)$ (assumed to be self-adjoint, i.e. $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$, in the complex case) that separates points (i.e. for every $x, y \in K$ with $x \neq y$ there is some $f \in \mathcal{A}$ such that $f(x) \neq f(y)$) and contains the unit $\mathbb{1}_K$, then $\mathcal{A}$ is dense in $C(K)$ with respect to the supremum norm.

Remark. By a classical result of measure theory, in the case where $\mathscr{F}$ is a σ-algebra the space $B(\mathscr{F})$ is just the space of all bounded $\mathscr{F}$-measurable scalar functions.

● **PROBLEM 4.6.** Suppose $\mathscr{F}$ is a set algebra having the subsequential completeness property (see Definition 6.9 from the lecture notes) and $(O_n)_{n=1}^\infty$ is a sequence of pairwise disjoint clopen subset of the Stone space of $\mathscr{F}$. Prove that there exists a subsequence $(O_{n_j})_{j=1}^\infty$ such that the set $\overline{\bigcup_{j=1}^\infty O_{n_j}}$ is open.

● **PROBLEM 4.7.** Let $\mathscr{F}$ be the algebra of all subset of $\mathbb{N}$ which are either finite or have finite complements. Give an example of a sequence $(\mu_n)_{n=1}^\infty$ of real-valued measures defined on $\mathscr{F}$ such that $\sup_n |\mu_n(E)| < \infty$ for each $E \in \mathscr{F}$, yet $\sup_n |\mu_n|(\mathbb{N}) = \infty$. In other words, show that the Nikodým Boundedness Principle (Theorem 2.5 from the lecture notes) fails for set algebras.

● **PROBLEM 4.8.** A series $\sum_{n=1}^\infty x_n$ in a Banach space X is called *weakly unconditionally Cauchy* (WUC for short) whenever $\sum_{n=1}^\infty |x^* x_n| < \infty$ for every $x^* \in X^*$. Give an example of a WUC series in c_0 that is not weakly convergent.

Remark. The example which you are supposed to find is, in a sense, the 'unique' example of a WUC series that is not weakly convergent (it is even the 'unique' WUC series that is not unconditionally convergent). In fact, the following theorem holds: A Banach space X does not contain an isomorphic copy of c_0 if and only if every WUC series in X is unconditionally convergent. Given a divergent WUC series $\sum_{n=1}^\infty$ in X one may find a sequence of vectors of the form $y_n = \sum_{j=p_n}^{q_n} x_j$, where $1 \leqslant p_1 < q_1 < p_2 < q_2 < \ldots$, which is (in some precise sense) equivalent to the standard unit vector basis of c_0; see [J. Lindenstrauss, *Classical Banach Spaces*, vol. 1, Prop. 2.e.4] and also [F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, pp. 40–42].

● **PROBLEM 4.9.** Show that $\ell_1(\Gamma)$ is not WCG, whenever Γ is uncountable.

Hint. Have a look at Problem 2.5.

● **PROBLEM 4.10.** A Banach space X is said to have *Pełczyński's property* (V) whenever the following characterisation of weak compactness in X^* holds true: A set $K \subset X^*$ is relatively weakly compact if and only if

$$\lim_{n\to\infty} \sup_{x^* \in K} |x^* x_n| = 0 \quad \text{for every WUC series } \sum_{n=1}^\infty x_n \text{ in } X \tag{$*$}$$

(see Problem 4.8). Explain that Grothendieck's Theorem 3.6 means exactly that $C(K)$-spaces, for any compact Hausdorff space K, have the property (V).

Hint. In fact, you will need Grotendieck's theorem only to show that the condition $(*)$ is sufficient for K being relatively weakly compact. Observe (and prove) also that $(*)$ actually implies that K is bounded, which was assumed in Grothendieck's theorem because of a somehow weaker assumption. The necessity of $(*)$ may be proved directly without using any special machinery.

● **PROBLEM 4.11.** Prove that if X is a Grothendieck space, then X^* is weakly sequentially complete (that is, all weakly Cauchy sequences are weakly convergent).

Hint. It follows quite directly from the very definition of Grothendieck space (Definition 6.1).

● **PROBLEM 4.12.** Let $X_1, X_2, \ldots$ be Banach spaces. Show that for any $1 \leqslant p < \infty$ we have the isometric isomorphism

$$\Big(\bigoplus_{n=1}^{\infty} X_n\Big)_p^* \simeq \Big(\bigoplus_{n=1}^{\infty} X_n^*\Big)_q,$$

where $1/p + 1/q = 1$.

● **PROBLEM 4.13.** Show that $\ell_\infty \simeq L_\infty(0,1)$.

Hint. Apply the Pełczyński Decomposition Method. You are allowed to use the injectivity of $L_\infty(0,1)$ (which follows from the fact that $L_\infty(0,1)$ is a dual $C(K)$-space; see [F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, Prop. 4.3.8]).

Remark. This was proved by Pełczyński in 1958.

● **PROBLEM 4.14.** Let $\mathscr{F} \subset \mathcal{P}\mathbb{N}$ be the set algebra consisting of all those sets that are either finite or have finite complements. Consider the measure $\mu\colon \mathscr{F} \to \mathbb{R}$ given by

$$\mu(E) = \begin{cases} |E|, & \text{if } E \text{ is finite} \\ -|\mathbb{N} \setminus E|, & \text{if } \mathbb{N} \setminus E \text{ is finite.} \end{cases}$$

Of course, μ is not σ-additive. However, show that μ naturally induces a σ-additive measure on the algebra of all clopen subsets of the Stone space $\mathsf{St}(\mathscr{F})$.

Remark. We used the term 'σ-additive' in reference to a measure defined on a set algebra, not a σ-algebra. This means that the measure in question is supposed to satisfy the countable additivity condition for every sequence of pairwise disjoint measurable sets, provided that their union is still measurable. This exercise shows how to produce a σ-additive measure from a non-σ-additive one and it also gives an example of an unbounded σ-additive measure defined on an algebra. Recall that every σ-additive, scalar-valued measure defined on any σ-algebra must be bounded. More generally, every *strongly additive* vector measure defined on any algebra must also be bounded (see [J. Diestel, J.J. Uhl, *Vector Measures*, Cor. I.1.19]).

● **PROBLEM 4.15.** Suppose $X_1, X_2, \ldots$ are WCG Banach spaces and $1 \leqslant p < \infty$ or $p = 0$. Show that $(\bigoplus_{n=1}^\infty X_n)_p$ is also WCG. How this can be generalised for $1 < p < \infty$?

● **PROBLEM 4.16.** Let $\sum_{n=1}^\infty x_n$ be a series in a Banach space X that is convergent to some $x \in X$. Assume that it is also *unconditionally convergent*, that is, $\sum_{n=1}^\infty x_{\pi(n)}$ converges in X for every permutation π of $\mathbb{N}$. Prove that:

(i) $\sum_{k=1}^\infty x_{n_k}$ converges for every sequence $\{n_1 < n_2 < \ldots\} \subset \mathbb{N}$;
(ii) $\sum_{n=1}^\infty x_{\pi(n)} = x$ for every permutation $\pi\colon \mathbb{N} \to \mathbb{N}$;
(iii) $\sum_{n=1}^\infty x_n$ is WUC (see Problem 4.8).

● **PROBLEM 4.17.** Prove that $(\bigoplus_{n=1}^\infty \ell_\infty^n)_1^*$ contains a complemented subspace isometrically isomorphic to ℓ_1 (note that ℓ_∞^n stands for the n-dimensional Banach space equipped with the maximum norm).

Hint. Consider the operator $T\colon (\bigoplus_{n=1}^\infty \ell_\infty^n)_1 \to c_0$ given by $T((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty x_n$; then $T^*(\ell_1)$ is isometric to ℓ_1 (why?). Observe also that T has uniformly bounded right inverses L_k on $\ell_\infty^k \subset c_0$, for each $k \in \mathbb{N}$ (i.e. $TL_k = I_{\ell_\infty^k}$). Using a compactness argument try to define an operator $S\colon (\bigoplus_{n=1}^\infty \ell_\infty^n)_1^* \to \ell_1$ satisfying $ST^* = I_{\ell_1}$ with the aid of which you may define a desired projection from $(\bigoplus_{n=1}^\infty \ell_\infty^n)_1^*$ onto $T^*(\ell_1)$.

Remark. This was observed by W.B. Johnson in 1972. Note that this assertion gives two nice counterexamples. Firstly, the space $X = (\bigoplus_{n=1}^\infty \ell_\infty^n)_1$ is weakly sequentially complete, although X^{**} is not, as it contains ℓ_∞. Secondly, the space X^* is an ℓ_∞-sum of finite-dimensional spaces ℓ_1^n (see Problem 4.12), yet it is not a Grothendieck space (which could be expected to happen, since as we know from Grothendieck's Theorem 6.3 the space ℓ_∞ itself is a Grothendieck space).

● **PROBLEM 4.18.** In Problem 4.10 we defined Pełczyński's property (V) of a given Banach space X by requiring that the condition $(*)$ characterises weak compactness in X^*. Show that this property may be equivalently defined by saying that X satisfies (V) whenever for every Banach space Y every unconditionally summing operator $T\colon X \to Y$ is weakly compact.

Note that T is called *unconditionally summing* (or *unconditionally converging*), provided that $\sum_{n=1}^\infty T(x_n)$ is unconditionally convergent for every WUC series $\sum_{n=1}^\infty x_n$ in X.

Hint. Suppose the above condition holds true and fix any set $K \subset X^*$ satisfying $(*)$. Consider the operator $T\colon X \to \ell_\infty(K)$ given by $T(x)(x^*) = x^*x$ and show that it is unconditionally summing.

For showing that the characterisation from Problem 4.10 implies that every unconditionally summing operator on X is weakly compact, apply Gantmacher's Theorem 4.2.

Remark. It may be proved that $T\colon X \to Y$ is unconditionally summing if and only if it does not fix any copy of c_0 which means that there is no subspace Z of X isomorphic to c_0 such that $T|_Z$ is bounded below (see [D. Przeworska-Rolewicz, S. Rolewicz, *Equations in Linear Spaces*, Theorem 8.4]). This is a striking improvement of what we have said in Remark to Problem 4.8 concerning the 'unique' non-convergent WUC series (just consider $X = Y$ and $T = I_X$). The proof of this equivalence is not very difficult but it requires some basic knowledge on Schauder bases, in particular a 'sliding-hump' type selection result due to Bessaga and Pełczyński. In view of this result, our assertion gives another translation of the fact that $C(K)$-spaces satisfy (V). Namely, this is exactly what Pełczyński' Theorem 4.5 says: every unconditionally summing operator on a $C(K)$-space must be weakly compact.

● **PROBLEM 4.19.** Prove that if a Banach space X has the property (V), then X^* is weakly sequentially complete. Do it twice, using both of the two equivalent definitions of unconditionally summing operators which we have described in Problem 4.18 and the remark following it.

Hint. For any fixed weakly Cauchy sequence $(x_n^*)_{n=1}^\infty \subset X^*$ define an operator $T\colon X \to c$ by $T(x) = (x_n^* x)_{n=1}^\infty$. All you need to know is that T is weakly compact (because then every subset of $\{x_n^*\colon n \in \mathbb{N}\}$ is relatively weakly compact, as $x_n^* = T^* e_n^*$; elaborate the details). To this end, you should show that:

- *If $\sum_{n=1}^{\infty} x_n$ is a* WUC *series in X, then $\sum_{n=1}^{\infty} T(x_n)$ (unconditionally) converges*, when using the first-mentioned definition of unconditionally summing operators. In this case you may find useful Schur's Theorem 2.9.
- *The supposition that T is bounded below on some isomorphic copy of c_0 leads to a contradiction*, when using the second-mentioned definition of unconditionally summing operators. This does not require any auxiliary combinatorial tool, like Schur's theorem (which is, more or less, just an incarnation of Phillips' lemma), because it is already in there! As we have said, to prove the equivalence between two ways of defining unconditionally summing operators we need a 'sliding-hump' type argument.